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Teaching Energy Balance Using Round Numbers: A Quantitative Approach to the Greenhouse Effect and Global Warming

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Abstract

The idea of energy balance used to explain the greenhouse effect and global warming is often confusing for students, primarily because the standard quantitative analysis uses many constants and units. A "round" unit method is presented, which maintains the quantitative aspects of the standard analysis, but is much more intuitive for students.

1 Introduction

The topic of global warming is of current interest, and is covered in meteorology courses as well as in the atmospheric component of some physics and astronomy curricula. The idea of energy balance is used to explain the greenhouse effect and global warming, after introducing the blackbody radiation equations. It is then straightforward to derive the temperature a planet would achieve given a balance between the incoming solar radiation and the outgoing blackbody infrared radiation. One can then expand the analysis to include the effect of an atmosphere, including such concepts as radiation transmission, latent heat, and convection. Unfortunately, in introductory science classes the units used to measure energy and power (Joules and Watts) in these calculations can be daunting, leading students to make errors and resort to "plug-and-chug" techniques.

An alternative method of covering the material found in many introductory meteorology textbooks[1,2] uses "round" units to explain the energy balance. Using the round unit method, one supposes 100 units of solar radiation striking the Earth, and then outlines all of the factors which influence that radiation: reflections, interactions with the atmosphere, etc. The students gain a sense for how the energy balance works, but there is never a direct *quantitative* calculation of temperatures.

As an instructor of an introductory science course one normally has a choice between a *quantitative* approach to the greenhouse effect and global warming, complete with algebra that troubles students, or a *qualitative* approach where the energy balance concepts are clear but no actual temperature values can be obtained. In this article, I outline a method which bridges this gap: the energy balance is understood using round numbers, but quantitative temperatures can be calculated. The method is also useful for more advanced students, because it makes back-of-the-envelope calculations of certain quantities easier.

2 Energy Balance

The *effective radiating temperature* is defined to be that temperature an object will reach when a balance is achieved between the sunlight striking the object and the blackbody radiation released by the object. The basic balance equation is simply

$$\text{(total energy in)} = \text{(total energy out)}$$

The temperature adjusts until this balance is achieved. If the object is receiving more energy than it is emitting, then the temperature increases, bringing the system closer to balance. Likewise, if the object is too hot and emits more than it receives, then the temperature will decrease back to equilibrium. The particular balance equations become more complicated when we include reflections and partial transmissions, as well as multiple sinks and sources of energy, but the basic concept is the same: the system adjusts until all parts of the system are in energetic balance.

Sunlight striking the Earth is usually expressed in terms of *solar irradiance*, or solar power per surface area. This quantity, also called the solar constant, has a value of about 1390 W / m ^{2}. Likewise, blackbody (or thermal) radiation released by the Earth is expressed in terms of *thermal irradiance*, also radiative power per surface area. Because the total energy per second striking the surface, or being released by the object, is the irradiance multiplied by the surface area we can rewrite the energy balance equation in *round units*, which are proportional to the irradiance[†] .

$$\text{(irradiance in)} \quad = \quad \text{(irradiance out)} \tag{1}$$

In these round units, we define the solar irradiance to be 100. The thermal irradiance values are dependent on the temperature of the object. Both experimental measurements and the quantum theory of blackbody radiation reveal that the total irradiance of an object increases as the temperature of the object raised to the fourth power, a relation known as Stefan's Law.[3] One can convert the standard blackbody equations to round units (derivation shown in Box 1), yielding the delightfully simple result:

$$\left(\frac{T}{88.5\text{ K}}\right)^4 \quad = \quad \text{irradiance units emitted (“round” units)} \tag{2}$$

These units are proportional to irradiance, although we have written them without explicit units, for simplicity. Using this equation, and the energy balance concept expressed in Equation 1, we can determine the temperature that the Earth would reach in a number of different scenarios.

We begin with the case of the Earth without an atmosphere, and assume a reflectance (or albedo) of the Earth of $A = 30\%$. The incoming solar irradiance is defined to be 100 round units, and the irradiance must balance both at the surface, and as observed from space (see Figure 1). We define the thermal irradiance emitted from the surface as E_{Surface}, and balance the incoming and outgoing radiation.

$$\text{(incoming)} \quad = \quad \text{(outgoing)}$$

$$\text{At the surface}$$

$$100 - \underbrace{30}_{\text{reflected}} \quad = \quad E_{\text{Surface}}$$

$$\text{From space}$$

$$100 \quad = \quad \underbrace{30}_{\text{reflected}} + E_{\text{Surface}}$$

which implies

$$\begin{aligned} E_{\text{Surface}} &= 70 \\ \left(\frac{T_e}{88.5\text{ K}}\right)^4 &= 70 \\ T_e &= 256\text{ K} \end{aligned} \tag{3}$$

Thus, without an atmosphere, the Earth would reach a temperature of 256 K. The standard method for performing these calculations is made more difficult with the many constants and units involved. Including the effects of an atmosphere becomes much more difficult for many students using the standard method rather than the round unit method. We present the case of an added atmosphere in the next section, using the round unit method. The standard quantitative treatment of this case is shown in Box 2.

[†] Because the Sun only strikes one half of the Earth, and the Earth emits radiation over its entire surface, the proportionality constant between the round units and standard units of irradiance are not the same for solar and thermal irradiance.

3 Including an Atmosphere: the Greenhouse Effect

The actual temperature of the Earth is around 288 K. In order to better approximate the Earth we need to include the effect of an atmosphere. To a first approximation we assume that the atmosphere is transparent to all (mostly visible) light from the Sun and absorbs all of the infrared emission emitted from the surface, as shown in Figure 2. The atmosphere heats to a temperature, T_a, and re-radiates energy, the irradiance denoted by E_{Atm}, both up into space and back down to the surface. The surface and atmosphere heat up until the balance of irradiance occurs at *every* stage of the system: on the surface, in the atmosphere, and from space. The energy balance equations are given below, and shown graphically in Figure 2.

$$\begin{array}{rcl}
\text{(incoming)} & = & \text{(outgoing)} \\
 & \text{From space} & \\
100 & = & 30 + E_{\text{Atm}} \\
 & \text{In the atmosphere} & \\
E_{\text{Surface}} & = & E_{\text{Atm}} + E_{\text{Atm}} \\
 & \text{At the surface} & \\
70 + E_{\text{Atm}} & = & E_{\text{Surface}}
\end{array}$$

These equations lead to $E_{\text{Atm}} = 70$ and $E_{\text{Surface}} = 140$. Again, to calculate temperatures we use

$$\left(\frac{T_a}{88.5\ \text{K}}\right)^4 = 70$$

$$\left(\frac{T_e}{88.5\ \text{K}}\right)^4 = 140$$

$$T_a = 256\ \text{K} \tag{4}$$

$$T_e = 304\ \text{K} \tag{5}$$

The effect of the atmosphere is to warm the surface, in what is called the greenhouse effect. Notice that the temperature that the atmosphere converges to, $T_a = 256$ K , is identical to the surface temperature in the no-atmosphere case (Equation 3). The assumption of 0% transmission of infrared through the atmosphere, and the omission of energy going to latent heat (evaporation of the oceans) causes the resulting surface temperature estimate to be higher than the measured surface temperature.

These balance equations we have discussed can be seen graphically in Figure 2. The simplicity of the method, especially with an easy graphical visualization, makes this method very useful in conveying difficult energy balance concepts.

4 Discussion

One can expand this analysis to include other planets in our solar system by adjusting the incoming solar irradiance, E_{Sun}:

$$E_{\text{Sun}} = \frac{100}{\text{distance}^2\ \text{(AU)}} \tag{6}$$

where 1 A.U. (astronomical unit) is the distance from the Earth to the Sun. One can also include other details of the energy transport such as latent heat, different amounts of absorption of infrared and visible, effects of clouds, etc.

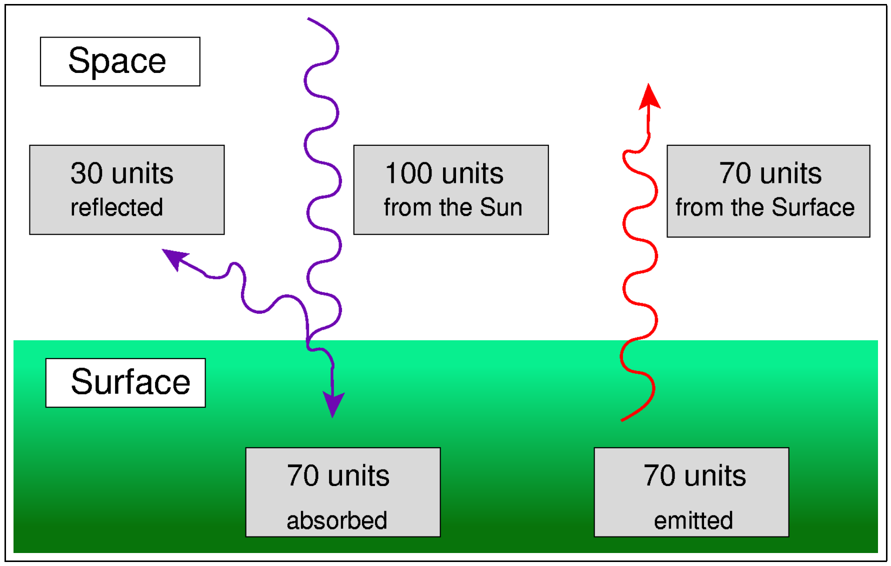

Figure 1: Energy Balance of the Earth without an Atmosphere. Shown are the round irradiance units for the incoming solar radiance (100 units), with 30 percent reflected (30 units), and the remaining 70 units absorbed by the surface. To balance the incoming energy, the surface heats until the thermal irradiance emitted is equal to 70 units. Balance is achieved at all levels of the system: from space there are 100 units entering the system and 100 units leaving, and from the surface there are 70 units incoming and 70 units outgoing.

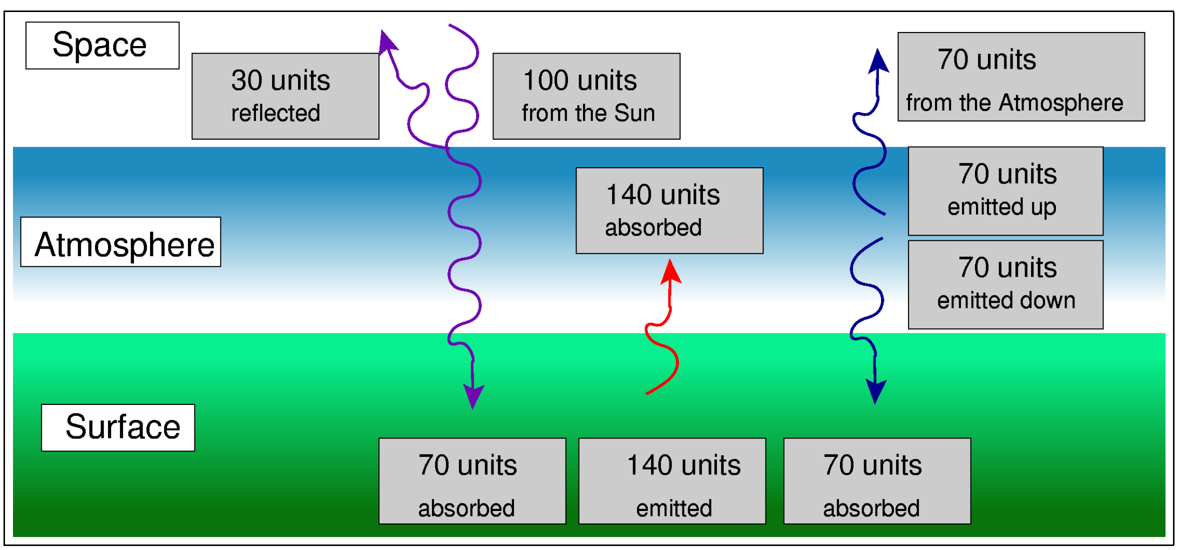

Figure 2: Energy Balance of the Earth with an Atmosphere. Shown are the round irradiance units for the incoming solar radiance (100 units), with 30 percent reflected (30 units) by the atmosphere, and the remaining 70 units passing through the atmosphere and absorbed by the surface. To balance the incoming energy, the surface heats until the thermal irradiance emitted is equal to 140 units, to balance both the incoming solar irradiance and the incoming thermal irradiance emitted by the atmosphere. Balance is achieved at all levels of the system: from space there are 100 units entering the system and 100 units leaving, from the atmosphere there are 140 units entering (from the surface) and 140 leaving (70 up, and 70 down), and from the surface there are 140 units incoming (70 from solar irradiance, and 70 from atmospheric thermal irradiance) and 140 units outgoing.

A quick example demonstrates the flexibility of the method. The real atmosphere has about a 10% transmission of infrared through the atmosphere, instead of the 0% transmission assumed in the previous examples. The balance would then look like Figure 3A, and the resulting balance equations would be

$$\begin{array}{rcl} \text{(incoming)} & = & \text{(outgoing)} \\ & & \\ & \text{From space} & \\ 100 & = & 30 + E_{\text{Atm}} + 0.1 \times E_{\text{Surface}} \\ & & \\ & \text{In the atmosphere} & \\ 0.9 \times E_{\text{Surface}} & = & 2 \times E_{\text{Atm}} \\ & & \\ & \text{At the surface} & \\ 70 + E_{\text{Atm}} & = & E_{\text{Surface}} \end{array}$$

Solving, we get $E_{\text{Atm}} = 57.3$ and $E_{\text{Surface}} = 127.3$, shown graphically in Figure 3B. These equations are both intuitive (from the picture) and easy to solve algebraically. To obtain the temperatures we use

$$\begin{aligned} \left(\frac{T_a}{88.5\text{ K}}\right)^4 &= 57.3 \\ T_a &= 243\text{ K} \\ \left(\frac{T_e}{88.5\text{ K}}\right)^4 &= 127.3 \\ T_e &= 297\text{ K} \end{aligned}$$

The effect of this "infrared hole" is (intuitively) a cooling effect, resulting in a temperature difference of about 7 degrees. Global warming is the filling of this "hole" with greenhouse gases (in addition to possible feedback loops), causing a temperature increase. With this method, it is straightforward to talk quantitatively about the effects of atmospheres and pollutants, yet avoid many of the difficulties the students have with the standard analysis. This method can also help in back-of-the-envelope calculations because of its very simple form. I have found it very useful in communicating difficult ideas to my students, and providing a method that they can use to achieve a *quantitative* understanding of atmosphere energy balance.

References

[1] Aguado and Burt. *Understanding Weather and Climate*. Prentice Hall, 2 edition, 2001.

[2] C. Donald Ahrens. *Meteorology Today: An Introduction to Weather, Climate, and the Environment*. Brooks/Cole, 7 edition, 2003.

[3] David Halliday, Robert Resnick, and Jearl Walker. *Fundamentals of Physics*. John Wiley and Sons, 6 edition, 2002.

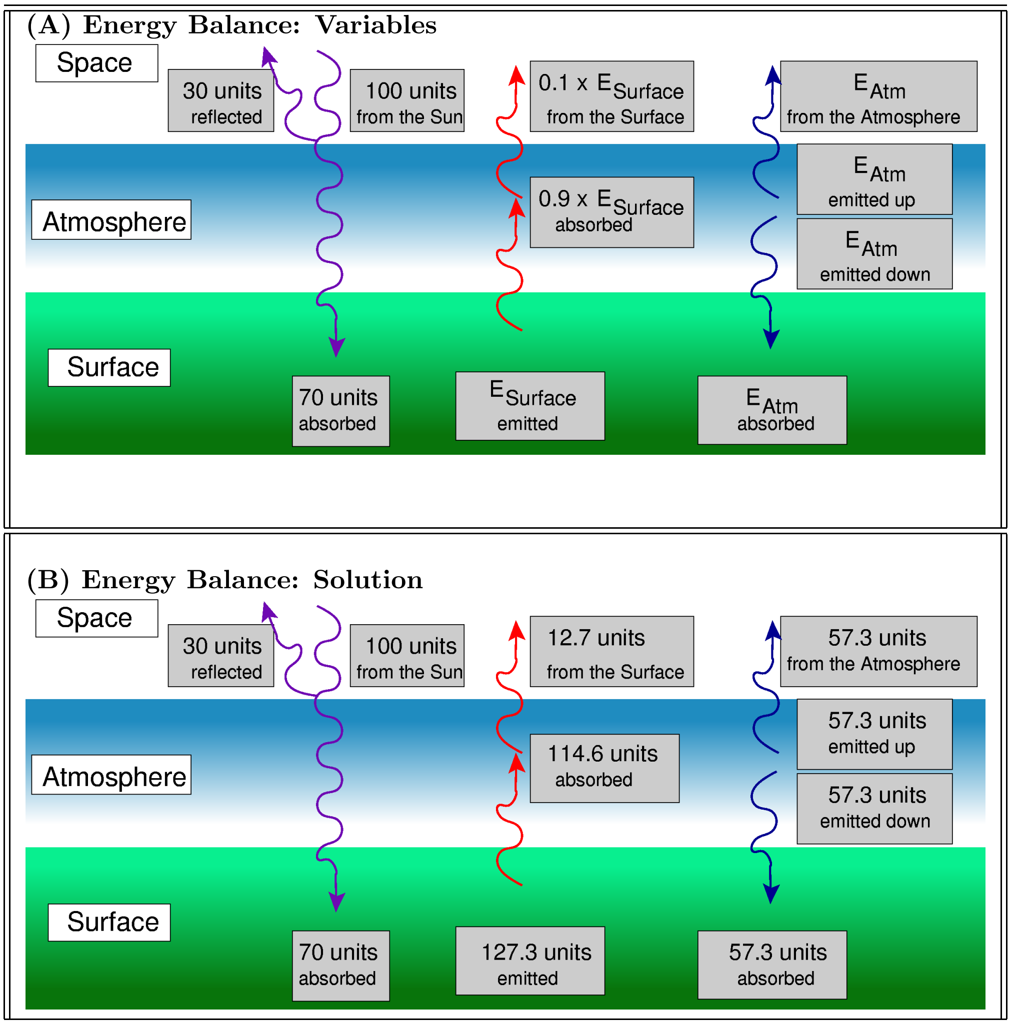

Figure 3: Energy Balance of the Earth with an Atmosphere with a Partial Transparency to Thermal Radiation. Shown in (A) are the same variables as in Figure 2, except that the atmosphere absorbs only 90% of the thermal radiation from the surface. The final solution to the balance is shown in (B). One can see that at every level (i.e. from space, the atmosphere, or the surface), the total incoming radiation balances the total outgoing radiation.

Box 1: Derivation of the Round Unit Equation

In this section we will derive the round-unit temperature equation (Equation 2) by solving the energy balance equations in the simplified case of an Earth without an atmosphere, and no reflectance.
The solar irradiance striking the Earth is approximately $S = 1390 \frac{\text{W}}{\text{m}^2}$. The total energy received by the Earth every second from the Sun is found by multiplying the solar irradiance by the cross-sectional area of the Earth (this takes into account changes in the incident angle of the sunlight, averaged over the Earth's surface).

$$\text{(total energy per second in)} = S \times \pi R_e^2$$

where R_e is the radius of the Earth.
The energy emitted per second from the entire surface of the Earth is given by the thermal irradiance (from Stefan's Law) multiplied by the surface area of the Earth

$$\text{(total energy per second out)} = \left(\sigma T_e^4\right) \times 4\pi R_e^2$$

where $\sigma = 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2\ \text{K}^4}$ is a measured constant in Stefan's Law and T_e is the temperature of the Earth.
Equilibrium is established when the incoming energy is equal to the outgoing energy, or

$$\begin{aligned} \text{(total energy per second in)} &= \text{(total energy per second out)} \\ S \times \pi R_e^2 &= \left(\sigma T_e^4\right) \times 4\pi R_e^2 \end{aligned}$$

Solving for the temperature we get

$$\begin{aligned} T_e^4 &= \frac{S}{4\sigma} = \frac{1390 \frac{\text{W}}{\text{m}^2}}{4 \times 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2\ \text{K}^4}} \\ &= 6.13 \times 10^9\ \text{K}^4 \\ T_e &= 280\ \text{K} \end{aligned} \tag{7}$$

To convert this to round units we ignore the surface area (which cancels anyway), and work directly with the irradiance. We assume 100 units of incoming solar irradiance, which is balanced by 100 units of emitted thermal irradiance from the Earth. The resulting equation must obey the temperature-to-the-fourth form of Stefan's Law

$$\left(\frac{T_e}{\text{(constant)}}\right)^4 = 100 \text{ round units}$$

and yield the same temperature as Equation 7

$$\begin{aligned} T_e^4 &= 100 \text{ round units} \times \text{(constant)}^4 = \frac{S}{4\sigma} \\ \text{(constant)} &= \left(\frac{S}{400\sigma}\right)^{1/4} = 88.5\ \text{K} \end{aligned}$$

Technically the units for the constant should be more complicated, in order to make the round units have dimensions of irradiance, but it is easier to leave the round "units" without units, and recall that they are simply proportional to irradiance.

Box 2: The Standard Quantitative Method

In this section we outline the standard quantitative derivation of the energy balance in the case of an Earth with an atmosphere which reflects 30% (albedo $A = 0.3$) of the incoming solar radiation and absorbs all of the thermal radiation from the surface. The same balance equations and notation used in Box 1 will be used. We introduce the temperatures of the atmosphere and the Earth's surface, T_a and T_e respectively. Compare this to the derivation done in Section 3 using round units.

We balance the total energy per second striking the surface, both from the solar radiation and the atmospheric thermal radiation. We also balance the incoming and outgoing radiation in the atmosphere and from space. The energy balance equations are

$$\text{(total energy in)} = \text{(total energy out)}$$

$$\textbf{From space:} \quad \underbrace{S \times \pi R_e^2}_{\text{solar energy in}} = \underbrace{A \times (S\pi R_e^2)}_{\text{reflected}} + \underbrace{\sigma T_a^4 \times 4\pi R_e^2}_{\text{from atmosphere}}$$

$$\textbf{In the atmosphere:} \quad \underbrace{\sigma T_e^4 \times 4\pi R_e^2}_{\text{from the surface}} = \underbrace{\sigma T_a^4 \times 4\pi R_e^2}_{\text{emitted upward}} + \underbrace{\sigma T_a^4 \times 4\pi R_e^2}_{\text{emitted downward}}$$

$$\textbf{At the surface:} \quad \underbrace{(1-A)S \times \pi R_e^2}_{\text{solar energy in}} + \underbrace{\sigma T_a^4 \times 4\pi R_e^2}_{\text{from atmosphere}} = \underbrace{\sigma T_e^4 \times 4\pi R_e^2}_{\text{emitted into atmosphere}}$$

which simplifies to

$$T_a^4 = \frac{(1-A)S}{4\sigma} = \frac{(1-0.3) \times 1390 \text{ W / m}^2}{4 \times 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2 \text{ K}^4}} = 4.29 \times 10^9 \text{ K}^4$$

$$T_e^4 = 2 \times \frac{(1-A)S}{4\sigma} = 2 \times \frac{(1-0.3) \times 1390 \text{ W / m}^2}{4 \times 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2 \text{ K}^4}} = 8.58 \times 10^9 \text{ K}^4$$

We get an atmospheric temperature of $T_a = 256$ K and a surface temperature of $T_e = 304$ K. It is interesting to see that the temperature that the atmosphere converges to, $T_a = 256$ K, is identical to the surface temperature in the no-atmosphere case (Equation 3).

Notice that one has to be keenly aware of the constants in the algebra, and their somewhat messy units, as well as numbers which can only easily be manipulated with a calculator. The round unit method eliminates all of these complications, and allows one to focus on the physics of the problem.